

Optimal Life Insurance Demand with Endogenous Wage Depending on Human Capital Factors

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Abstract

We develop a firm-household integrated model in which the firm hires the household wage earner as a technical labor by committing to a long-term wage contract, and the household receives wages facilitating its dynamic wealth allocation with life insurance purchase. The firm assesses the value contribution of labor human capital with allowing for human-capital-specific factors. Endogenous wage takes an explicit structural form, derived from the value-sharing rule between shareholders and labor. The optimal wealth allocation choices are solved using dynamic programming principle. We find that heterogeneities in human capital factors shape the across-household differences in life insurance holding via endogenous wage. Households possessing a higher quality of human capital exhibit a higher demand for life insurance, while human capital depreciation, substitutability, and performance risk are negative factors discouraging households' insurance purchase.

Keywords: Human Capital Factors; Endogenous Wage; Life Insurance; Wealth Allocation.

JEL classification: C61; G11; G52; J24; J30.

1. Introduction

The role of human capital in forming households' life insurance demand has received much attention from research in finance and economics. Hedging wealth losses due to premature death is regarded as a human-capital-protection motive for households to enter life insurance contracts. Higher return on human capital (income) motivates households to spend more in purchasing life insurance (see Pliska and Ye, 2007; Chen et al., 2006; or Chen et al., 2022). The existing literature regarding optimal insurance demand narrowly focuses on these two dimensions of human capital. Many previous attempts in fact propose a variety of human capital factors that affect labor share in firms and might generate the across-household differences in life insurance holding through labor wages/income. For example, Bai and Wang (2003) point out that under some circumstances, labor-productivity uncertainty (as a type of performance risk on human capital) causes lower wages and discourages household investment in specific human capital. Similarly, Tang et al. (2024) find an inverse association between human capital risk and employee wage/compensation in state-owned companies. Fairris and Bujanda (2008) and Dao et al. (2019) document the detrimental influence of labor substitutability on workers' wage gain or income share. Acemoglu and Shimer (2000) find that firms adopting better productivity technologies (implying a higher human capital quality) pay more wages to workers. Hombert and Matray (2023) provide evidence of long-run wage discount arising from human capital depreciation. These findings motivate us to explore how human capital factors influence household life insurance demand.

We develop a firm-household integrated model in which the firm pays wages according to the value-sharing rule between shareholders and technical labor (i.e., household wage earner) and the household receives wages (from the firm) facilitating its dynamic wealth allocation among life insurance purchase, consumption, financial investment, and savings. The proposed model features endogenous wages solved from the value-sharing rule, which enables us to capture the influence of changes in human capital factors on firm value increments contributed by labor human capital. These factors include human capital quality, depreciation, performance risk, and substitutability. In contrast, the existing life-cycle models of life insurance demand largely rely on the setting of exogenous wages, and thus, fail to capture the effect of human capital factors. Besides, our model

features a closed-form solution to dynamic wealth allocation strategy. This permits us to analyze related comparative statics in a parsimonious and tractable way.

In the model, we treat the labor's productivity technology as the quality of his human capital and specify the dynamics of productivity technology by using a standard diffusion process. This specification offers a useful guide to measuring human capital depreciation and performance risk from the expected decaying rate of technology and uncertainty over productivity technology. Also, we employ the pay-performance ratio specified in the employment contract as an inverse measure of human capital substitutability. In the specification of this ratio, *pay* refers to the total present value of wage flows promised by the employment contract, while *performance* refers to marginal increments in the expected firm value generated by the technological advantages of labor human capital. The pay-performance ratio, which embodies the value-sharing rule, reflects labors' power of bargaining for wages. A strong bargaining power intuitively means that the labor might possess scarce human capital and is non-substitutable (the elasticity of the labor-labor substitution is low). Therefore, when the degree of human capital substitutability is higher, the household and the firm may agree on a lower pay-performance ratio.

We solve the structural form of endogenous wages consisting of the aforementioned human capital factors, and then merge it into the framework of dynamic wealth allocation proposed by Pliska and Ye (2007). Our numerical results show that endogenous wages can be a critical channel through which heterogeneities in human capital factors generate the across-household variations in life insurance demand. We find human capital quality to be a factor that influences household life insurance purchase positively, while human capital depreciation, risk, and substitutability are negative factors discouraging insurance purchase. This is because when human capital possessed by the labor in place has a high quality (provides unstable performance, depreciates quickly, and is highly substitutable), the expected value contribution from human capital is large (the expected value contribution is small, the pay-performance ratio is low) and firms set a high (low) level of wage according to the value-sharing rule. Given this fact, households correspondingly spend more (less) in purchasing life insurance.

Our paper is related to the vast literature on the life-cycle theory of household life insurance demand. Prior studies focus on the impact of planning horizon uncertainty (Yaari, 1965; Richard, 1975; Pliska and Ye, 2007), financial investment (Campbell, 1980; Pirvu and Zhang, 2012), family members (Lewis, 1989; Kwak et al., 2011), preferences/attitude (Huang et al., 2008; Jensen and Steffensen, 2015; Birghila et al., 2023; Li et al., 2023; Peng and Li, 2023; Steffensen and S  e, 2023; Wang et al., 2024), un-spanned income loss in unrecoverable health shocks (Hambel et al., 2017), inflation risk (Kwak and Lim, 2014; Han and Hung, 2017), financial constraint (Zeng et al., 2016), surrender option guarantee (Kronborg and Steffensen, 2015), environmental risk (Shen and Wei, 2016), the distinction between systematic and idiosyncratic mortality risks (Shen and Sherris, 2018), life settlement (Seog and Hong, 2019), and financial market incompleteness/frictions (Liu et al., 2021; Chen et al., 2022). We depart from the line of this literature by extending research in the dimension of human capital factors.

In addition to research scope, another difference between previous studies and ours lies in the specification of wages. As mentioned above, the literature largely adopts the setting of exogenous wages that treats initial income/wage as an arbitrary constant. In contrast, we build a firm-based model of endogenous wages and solve the structural formula for wages from the value-sharing rule. This formula allows us to complement the literature by capturing the implications of human capital factors for household life insurance purchase. These implications offer several new insights from heterogeneities in human capital quality, depreciation, performance risk, and substitutability for explaining the across-household differences in life insurance holding. Besides, our paper makes a methodological contribution by proposing an integrated framework of household and corporation. Our framework can be viewed as the first step toward bridging the gap between corporate finance issues and household life insurance demand theories (via endogenous wage). Numerous empirical studies document the impact of corporate financial policies (e.g., debt use, payout, cash holding, R&D investment, etc.) on labor cost (Klasa et al., 2009; Chino, 2016; Graham et al., 2023; Howell and Brown, 2023; and others). Such empirical findings and our framework can jointly inspire subsequent works to advance the theories of insurance demand in corporate-finance dimensions.

Also, our paper is related to a long list of empirical studies identifying the determinants of household life insurance ownership (see, e.g., Browne and Kim, 1993; Hau, 2000; Li et al., 2007; Gutter and Hatcher, 2008; Lee et al., 2010; Liebenberg et al., 2012; Sliwinski et al., 2013; Fier and Carson, 2015; Outreville, 2015; Luciano et al., 2016; Zerriaa and Noubbigh, 2016; Lin et al., 2017; and Wang et al., 2021). They have proposed various household-specific determinants (e.g., age, education, income, employment status, family structure, estate liquidity, risk aversion, the number of dependents, etc.) and systematic economic determinants (e.g., gender differences, catastrophes, financial literacy, racial differences, financial development, social security expenditures, inflation, national income, etc.). The testable implications generated by our model offer them a useful guide to finding new human capital factors for predicting life insurance holding, and help improve the explanatory power of empirical factor models for observed variations in life insurance holding.

2. An Integrated Model of Firm and Household

We consider a continuous-trading economy characterized by a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ satisfying the usual conditions. Ω denotes the state space, $\mathcal{F}_t$ denotes the set of information generated by triple-dimensional standard Brownian motions $[w_t, b_t, z_t]^T$, and $\mathbb{P}$ is the real measure. Time continuously varies over $[0, \infty)$. In this economy, a representative household wage earner works as a technical labor for the representative totally-equity-financed firm engaging in producing and selling products.^① This firm determines the level of wages to the household by considering the factors of labor human capital (including quality, risk, depreciation, and substitutability). The household receives wages from the firm, which facilitate its decisions on dynamic wealth allocation among consumption, financial investment, life insurance purchase, and private savings. Below we construct Figure 1 to facilitate understanding the structure of our firm-household integrated framework.

^① While several empirical studies (Hanka, 1998; Matsa, 2010; Chemmanur et al., 2013; Graham et al., 2023; etc.) document the significance of corporate debt usage in explaining the dispersion of employee costs in the cross section, we find no evidence on the human-capital-based mechanism through which corporate debt issuance influences wage payments to employees. Besides, if our model allows for the effect of corporate debt, it will complicate the model but fail to generate additional economic insights, since debt use does not alter the influence of human capital factors on endogenous wages qualitatively. These facts motivate us to adopt the setting of zero debt in the wage model.



Figure 1: Concept of the Firm-Household Integrated Framework.

2.1. Firm-Side Model Setup

Consider a circumstance where the representative firm attempts to hire a technical labor in charge of production (operates the mechanized production system or maintains its operation).

2.1.1. Production function

As in Miao (2005), the production function $\mathcal{F}: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ takes the form of $\mathcal{F}(k_t) \equiv k_t^\gamma$, where $\gamma \in (0,1)$ is the return-to-scale parameter and k_t is the input at time t with a fixed unit cost $\delta > 0$ set by the firm's supplier.

2.1.2. Productivity technology on human capital

If the technical labor is in place, productivity technology on his human capital will evolve according to the following diffusion process

$$dA_t = \mu_A A_t dt + \sigma_A A_t dw_t, \quad A_0 \equiv \mathcal{A} \quad (1)$$

where A embodies the quality of human capital, $\mu_A < 0$ denotes the depreciation rate of human capital, and $\sigma_A > 0$ captures uncertainty on human capital performance (human capital risk).^②

2.1.3. Product demand

We follow Pichler et al. (2008) and express the firm's aggregate product demand as

^② A negative drift rate of the specification of productivity technology suggests that the quality of human capital will depreciate over time. Given such a notion, we employ $\mu_A < 0$ as a measure of the human capital depreciation rate.

$$\hat{Q}_D(p_t, Q_t) \equiv q(p_t) Q_t \equiv p_t^{-\varepsilon} Q_t \quad (2)$$

where p_t is product price, $q(\cdot)$ is the size of customer base, $\varepsilon \in (1, 1+\gamma^{-1})$ is the customer -to-price elasticity, and Q_t is the average demand per customer governed by a diffusion process

$$dQ_t = \mu_q Q_t dt + \sigma_q Q_t d\delta_t, \quad Q_0 \equiv Q \quad (3)$$

In (3), μ_q is the growth rate and σ_q is the volatility rate. The product demand dynamics are not affected by fluctuations in productivity technology (because these fluctuations never alter product features/functionality), and so, the two Brownian motions, δ and w , are mutually independent.

2.1.4. *Employment contract*

We follow Berk et al. (2010) and assume the firm's managers employ the household wage earner as a technical labor by committing to a long-term contract. This contract continuously pays wages $I_t = I e^{gt}$ where I is the endogenously-determined wage level at initial time and g is the predetermined wage growth rate. The wage payments are totally financed by selling additional equity. The contract is terminated at the wage earner's retirement date T or the date of his death τ if he dies before retirement (the specification of mortality risk will be given in Section 2.2.1). The wage level is determined according to a fixed pay-performance ratio $\beta \in [0, 1]$ that embodies the value-sharing rule between shareholders and labor.^③ A higher pay-performance ratio suggests that the technical labor has a stronger power of bargaining for wages, and hence, implies a lower degree of human capital substitutability. In the specification of this ratio, *performance* refers to

^③ The concept of the value-sharing rule is inspired by and similar to the surplus-sharing rule in Michaels et al. (2019). That article quantifies workers' bargaining power using the ratio of worker surplus to the sum of worker surplus and firm surplus. Therefore, the pay-performance ratio in our model can be thought of as an alternative proxy for labors' bargaining power or inverse measure of labor substitutability. The cross-sectional variations in the pay-performance ratio is attributed to the influence of permanent fluctuations in labor demand and supply on labor market competition. Issues related to labor market dynamics (e.g., labors' competitive behavior in job seeking, the impact of human capital scarcity on employment decisions, human capital allocation in industries, etc.) are beyond the scope of our research, and thus we consider an exogenous value-sharing rule and assume the pay-performance ratio to be stationary, which exogenizes the influence of labor market competition and bargaining on the employment contract design as a given requirement for employment.

increments in the expected firm value generated by labor human capital, while pay refers to the total present value of wage flows promised by the employment contract. The contract requires the technical labor to give up all outside options. Agency frictions between labor and managers do not exist, and hence, the value-sharing rule ensures that labor always makes best effort in working.^④

2.1.5. Operating earnings and expected firm value

Instantaneous operating earnings at an arbitrary time t are given by:^⑤

$$\hat{\pi}(p_t, Q_t, A_t) = \underbrace{p_t \hat{Q}_D(p_t, Q_t)}_{\text{sales revenue}} - \underbrace{\delta k^*(p_t, Q_t, A_t)}_{\text{production costs}} \quad (4)$$

where $k^*(p_t, Q_t, A_t) = p_t^{-\varepsilon/\gamma} Q_t^{1/\gamma} A_t^{-1/\gamma} = k_t^*$ is the equilibrium demand for inputs and is derived from the market-clearing condition $\hat{Q}_D(p_t, Q_t) = \hat{Q}_S(A_t, k_t^*) \equiv A_t \mathcal{F}(k_t^*)$. Note that the outlays for wage payments are excluded from the calculation of operating earnings, because such outlays are externally financed from new stock issuance.

Hence, the expected value of firm assets conditional on current employment takes a standard form of the total present value of future earnings flows:

$$V(p_t, Q_t, A_t) = \mathbb{E}_t \int_t^{T \wedge \tau} \hat{\pi}(p_s, Q_s, A_s) e^{-r(s-t)} ds \quad (5)$$

where $\mathbb{E}_t$ is the expectation operator conditional on $\mathcal{F}_t$ and r denotes the riskless interest rate (shareholders are risk-neutral and discount future cash flows at the riskless interest rate).

Also, we consider a stationary industrial benchmark of productivity technology A , which helps us measure increments in the expected firm value generated by labor human capital. Such increments are measured using $V(p_t, Q_t, A_t) - V(p_t, Q_t, A)$.

^④ Prior studies on agency theories typically consider the agency conflict between managers and shareholders or debt holders (e.g., Leland, 1998; Ju and Wan, 2012; and Huang et al., 2023), but pay less attention to the conflict between managers and employees. Moreover, evidence on information asymmetry between managers and employees is little in the empirical literature. Given these facts, we assume that managers can monitor employees' performance as well as working effort costlessly, and that the problems of agency conflicts between them do not exist.

^⑤ The fixed operating cost plays no role in solving the optimal product price, and does not create new insight to our research. Motivated by this fact, we follow Miao (2005) and assume the fixed operating cost to be zero.

2.1.6. Decision problem

When considering product pricing strategies, firm managers' objective is to continuously sell products at the price that maximizes instantaneous operating earnings:

$$\max_{p_t \geq 0} \hat{\pi}(p_t, Q_t, A_t). \quad (6)$$

Plugging the optimal product price into (4) yields operating earnings under optimal product pricing: $\hat{\pi}^*(Q_t, A_t) \equiv \hat{\pi}(p_t^*, Q_t, A_t)$ where $p_t^* \in \arg \max \hat{\pi}(p_t, Q_t, A_t)$. Correspondingly, the expected firm value and its increments due to labor human capital can be re-expressed by

$$V^*(Q_t, A_t) \equiv V(p_t^*, Q_t, A_t) \text{ and } Perf(t; Q_t, A_t) = V^*(Q_t, A_t) - \tilde{V}(Q_t, A)$$

where $\tilde{V}(Q_t, A) \equiv V(\tilde{p}_t, Q_t, A)$ and $\tilde{p}_t \in \arg \max \hat{\pi}(p_t, Q_t, A)$. Using the pay-performance ratio and $Perf(\cdot)$, managers determine initial wages I^* that satisfy the value-sharing rule:

$$\beta = Pay(0; I^*) / Perf(0; Q, A) \quad (7)$$

In (7), $Pay(0; I^*) \equiv \mathbb{E}_0 \int_0^{T \wedge \tau} I^* e^{(g-r)s} ds$ is the total present value of wage flows. Changes in the continuation value of the employment contract are jointly determined by the effect of mortality risk and the effect of wage growth. This continuation value decreases (increases) with time when the former (latter) effect dominates, and suddenly falls to zero if labor dies.

2.2. Household-Side Model Setup

Consider a circumstance where the household attempts to hedge against wealth losses due to mortality risk on human capital by purchasing life insurance when considering wealth allocation.

2.2.1. Mortality specification

The household wage earner, employed by the representative firm, has an uncertain lifetime

τ . This random time is specified using a one-jump counting process with the survivor function:

$$F(t) \triangleq \mathbb{P}(\tau \geq t) = \exp\left(-\int_0^t \lambda(u) du\right), \quad \lambda(u) = \tilde{\lambda} + \hat{\lambda}u \quad (8)$$

where $\lambda(\cdot)$ is a time-dependent deterministic hazard rate and both $\hat{\lambda}$ and $\tilde{\lambda}$ are fixed positive coefficients (as in Pliska and Ye, 2007). $\hat{\lambda}$ plays a role as initial hazard rate, while $\tilde{\lambda}$ captures the natural property that the hazard rate increases with time linearly. Although the time sensitivity of the hazard rate might be increasing or decreasing in time, for brevity, we omit this second-order time-varying effect. Also, we omit the dispersion of the hazard rate generated by heterogeneities in human capital factors, since no evidence on the associations between mortality risk and above-mentioned human capital factors is found by existing empirical studies.

The survivor function, in effect, helps measure the probability that the length of life is at least longer than t . The hazard rate $\lambda(t)$ represents the instantaneous death rate of the wage earner who has survived until time t , and satisfies the following property:

$$\lambda(t) \triangleq \lim_{dt \rightarrow 0} \frac{\mathbb{P}(t+dt > \tau \geq t \mid \tau \geq t)}{dt}.$$

Then the probability density function of the random death time can be given by

$$f(t) = -\partial F(t) / \partial t = \lambda(t) \exp\left(-\int_0^t \lambda(u) du\right) \quad (9)$$

Using (8) and (9) easily yields the explicit form of conditional death probability density at time t $f(s, t) = f(s) / F(t)$ and of conditional survival probability at time t $F(s, t) = F(s) / F(t)$.

2.2.2. Life insurance contract

Following Pliska and Ye (2007), we consider a representative insurance company that sells life insurance term contracts continuously. The length of this contractual term is infinitesimally small. The wage earner enters a life insurance contract by paying premiums ξ_t at each time point.

If the wage earner dies, then the insurance company pays an insurance amount $\xi_\tau / \alpha(\tau)$ where $\alpha : [0, T] \rightarrow \mathbb{R}^+$ is called as the premium-insurance ratio. Specifically, this ratio takes an explicit form: $\alpha(\tau) = \tilde{\alpha} + \hat{\alpha}\tau$ where $\tilde{\alpha}$ and $\hat{\alpha}$ are both fixed positive coefficients.

2.2.3. Financial investment opportunities

The financial market allows the wage earner to trade a well-diversified portfolio at the price S_t without suffering friction (i.e., no transaction costs, no selling constraints, and no tax-related duties).[®] Changes in the price of diversified portfolio evolve according to

$$dS_t = \mu_S S_t dt + \sigma_S S_t dz_t, \quad S_0 \equiv S \quad (10)$$

where $\mu_S > 0$ is the growth rate and $\sigma_S > 0$ is the volatility rate. These changes are independent of human capital performance (uncertainty over the firm's productivity technology can be thought of as a firm's idiosyncratic risk) but probably have a correlation with product demand. We do not specify this correlation in the present study, because it plays no role in examining the implication of human capital factors for wealth allocation and for life insurance demand. Given this fact, we assume the three Brownian motions (b , w , and z) are mutually independent.

2.2.4. Wealth allocation and dynamics

The wage earner continuously fulfills consumption plans c_t and pays premiums ξ_t to life insurance company by using his wealth X_t . Further, he allocates a portion of wealth to financial investment in the diversified portfolio θ_t , and remaining wealth to savings in his money market account $M_t = e^{rt}$. Hence, the wealth dynamics has the integral representation: on $t \leq \min(T, \tau)$,

$$X_t = \mathbf{x} - \int_0^t c_u du - \int_0^t \xi_u du + \int_0^t I_u^* du + \int_0^t \theta_u (dS_u / S_u) + \int_0^t (X_u - \theta_u) (dM_u / M_u) \quad (11)$$

[®] Several attempts have been made to examine the impact of market incompleteness arising from financial frictions on household dynamic wealth allocation (e.g., Zeng et al., 2016; Liu et al., 2021; and Chen et al., 2022). To the best of our knowledge, these studies never find evidence on market incompleteness due to human-capital heterogeneities. To restrict the attention on the scope of our research, hence, we consider the setting of a complete financial market.

where $X_0 \equiv \mathbf{x}$ is initial wealth and all wealth allocation strategies follow an $\mathcal{F}_t$ -adapted bounded process satisfying $\int_0^\infty c_u du < \infty$, $\int_0^\infty \xi_u du < \infty$, and $\int_0^\infty \theta_u du < \infty$.

2.2.5. Decision problem

The wage earner has a standard CRRA (constant relative risk aversion) preference and will discount future utility at the rate of ρ . His objective is to solve the decision problem on dynamic wealth allocation among consumption, financial investment, life insurance purchase, and savings:

$$\begin{aligned} \max_{(c, \xi, \theta) \in \mathcal{A}(X)} \mathbb{E}_0 \left[\int_0^{T \wedge \tau} U(c_t) e^{-\rho t} dt + U(Y_\tau) e^{-\rho \tau} 1_{(\tau \leq T)} + U(X_T) e^{-\rho T} 1_{(\tau > T)} \right] \quad (12) \\ \text{s.t. } dX_t = [rX_t - c_t - \xi_t + I_t^* + \theta_t(\mu_S - r)] dt + \theta_t \sigma_S dz_t. \end{aligned}$$

In (12), $U(x) \equiv \Gamma^{-1} x^\Gamma$ is the utility function, $\Gamma \leq 0$ is the risk-aversion parameter, $\mathcal{A}(X)$ is the set of all admissible decision strategies, and Y_τ is the total legacy left by the wage earner at the date of his death. The legacy includes his remaining wealth X_τ and compensation received from insurance company $\xi_\tau / \alpha(\tau)$. That means, $Y_\tau = X_\tau + \xi_\tau / \alpha(\tau)$.

3. Solving the Model

We solve the model in two stages. The first stage will derive the optimal product price from decision problem (6) and endogenous wages from the value-sharing rule (7). Plugging endogenous wages into decision problem (12), the second stage solves optimal wealth allocation strategy. For technical details about proofs for model solutions, please see appendices.

3.1. Stage 1: Product Pricing Strategy and Endogenous Wages

We derive the optimal product price from the objective of maximizing instantaneous earnings. Differentiating the earnings function (4) with respect to p_t , setting this expression being zero with $p_t = p_t^*$, and solving for p_t^* , we obtain the following proposition.

Proposition 1. (Optimal Product Pricing and Operating Earnings) *Given the earnings function (4), the firm sells products at $p_t^* = Q_t^{(1-\gamma)/\eta} A_t^{-1/\eta} \left(\frac{\delta\varepsilon}{\varepsilon-\eta}\right)^{\gamma/\eta}$. Instantaneous operating earnings under optimal product pricing takes the form of $\hat{\pi}^*(Q_t, A_t) = \mathbb{A}(Q_t, A_t) \Phi$ where $\eta \equiv \varepsilon + \gamma - \varepsilon\gamma$, $\mathbb{A}(a, b) \equiv a^{1/\eta} b^{(\varepsilon-1)/\eta}$, and $\Phi \equiv \left(\frac{\delta\varepsilon}{\varepsilon-\eta}\right)^{1-\varepsilon/\eta} \frac{\eta}{\varepsilon}$. The optimal product price increases with product demand ($\partial p_t^* / \partial Q_t > 0$) but decreases with technology ($\partial p_t^* / \partial A_t < 0$).*

Product pricing plays an important mechanism through which the firm absorb the benefit of technological advantages on labor human capital. If the technical labor achieves improvements in productivity technology, the firm can simultaneously attract more customers and raise earnings by cutting the product price. Such a pricing strategy enables the firm to convert intangible benefit on human capital performance into tangible earnings flows. Human capital benefits enjoyed by the firm decrease with the human capital depreciation rate, and are more volatile when human capital performance is riskier (since earnings volatility is increasing in human capital risk and the growth rate of earnings decreases with the human capital depreciation rate).

Consider next endogenous wages. Note that wage payments have a linear relation with the total present value of wage flows. We thus let $VF \equiv \mathbb{E}_0 \int_0^{T \wedge \tau} e^{(g-r)s} ds$ and have $Pay(\cdot) = I^* VF$. Then using the value-sharing rule (7) yields the proposition of endogenous wages below.

Proposition 2. (Endogenous Wages) *According to the value-sharing rule, managers set initial wages paid to the technical labor at*

$$I^* = \frac{\beta Perf(0; Q, \mathcal{A})}{VF}$$

where

$$\begin{aligned} Perf(0; Q, \mathcal{A}) &= \mathbb{B}(Q, \mathcal{A}, Q) - F(T, 0) e^{-rT} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbb{C}(s, u, T) e^{s/\eta} ds du \\ &\quad - \int_{0_+}^T \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(v, 0) \mathbb{C}(s, u, v) e^{s/\eta - rv} ds du dv, \\ VF &= \left[\mathbb{D}(T) F(T, 0) + \int_{0_+}^T \mathbb{D}(s) f(s, 0) ds \right] (g-r)^{-1}, \end{aligned}$$

$$\mathbb{B}(x, y, z) \equiv \mathbb{A}(x, y) K - \mathbb{A}(z, \mathcal{A}) G, \quad \mathbb{C}(x, y, z) \equiv f_0^{\ln Q}(x, z) f_0^{\ln A}(y, z) \mathbb{B}(1, e^y, 1),$$

$$\mathbb{D}(x) \equiv e^{(s-r)x} - 1, \quad \mathbb{E}(x, y, z) \equiv yx^{-1} + 0.5z^2(x^{-1} - 1)x^{-1},$$

$$\mathbb{F}(x, y) \equiv 0.5x^{-2}y^{-1}, \quad \mathbb{G}(a, b, c, d, e) \equiv \left[a - \ln b - (c - 0.5d^2)e \right]^2,$$

$$K \equiv \Phi \left[r - \mathbb{E}(\eta(\varepsilon - 1)^{-1}, \mu_A, \sigma_A) - \mathbb{E}(\eta, \mu_q, \sigma_q) \right]^{-1}, \quad G \equiv \Phi \left[r - \mathbb{E}(\eta, \mu_q, \sigma_q) \right]^{-1},$$

$$f_0^{\ln Q}(x, z) = \sqrt{\pi^{-1} \mathbb{F}(\sigma_q, z)} \exp \left[-\mathbb{G}(x, Q_t, \mu_q, \sigma_q, z) \mathbb{F}(\sigma_q, z) \right],$$

$$\text{and } f_0^{\ln A}(y, z) = \sqrt{\pi^{-1} \mathbb{F}(\sigma_A, z)} \exp \left[-\mathbb{G}(y, A_t, \mu_A, \sigma_A, z) \mathbb{F}(\sigma_A, z) \right].$$

Proposition 2 indicates that our model captures the influence of human capital factors on the wage payment. For example, the firm pays more wages to the technical labor when human capital substitutability is lower (the pay-performance ratio is higher). Human capital quality, depreciation, and risk influence the level of wages through the benefits of human capital performance enjoyed by the firm. These model features enable us to further examine how human capital factors affect the wage earner's life insurance demand and wealth allocation decision-making through wages.

3.2. Stage 2: Dynamic Wealth Allocation with Life Insurance Purchase

Given Propositions 1 and 2, we solve optimal wealth allocation strategies by using dynamic programming approach. The randomness of the wage earner's death is independent of uncertainty over his financial investment, and hence, we can rewrite decision problem (12) as the following:

$$\max_{(c, \xi, \theta) \in \mathcal{A}(X)} \mathbb{E}_0 \left[\int_0^T (F(t, 0)U(c_t) + f(t, 0)U(Y_t)) e^{-\rho t} dt + F(T, 0)U(X_T) e^{-\rho T} \right]$$

and define the indirect utility function

$$J(t, X_t) = \max_{(c, \xi, \theta) \in \mathcal{A}(X)} \mathbb{E}_t \left[\int_t^T (F(s, t)U(c_s) + f(s, t)U(Y_s)) e^{-\rho(s-t)} ds + F(T, t)U(X_T) e^{-\rho(T-t)} \right]$$

This implies the corresponding Hamilton-Jacobi-Bellman (HJB) equation (for brevity, we omit the time subscripts of wealth, wages, and all decision controls):

$$\max_{c, \xi, \theta} \left\{ J_t + J_X [rX - c - \xi + I^* + \theta(\mu_S - r)] + 0.5 J_{XX} \theta^2 \sigma_S^2 + \lambda(t)U(Y) + U(c) - [\lambda(t) + \rho]J \right\} = 0$$

subject to the boundary condition $J(T, X_T) = U(X_T)$. Using the HJB equation yields the first-order conditions of optimal wealth allocation strategies as well as their explicit solutions.

Proposition 3. (Wealth Allocation with Life Insurance Purchase) *Given the HJB equation of decision problem (12), the optimal rules of consumption, financial investment, and life insurance purchase (c^*, θ^*, ξ^*) satisfy the following three first-order conditions:*

$$U_c(c^*) - J_X = 0, \quad \theta^* J_{XX} \sigma_S^2 + J_X(\mu_S - r) = 0, \quad \text{and} \quad \frac{\lambda(t)U_Y(Y(\xi^*))}{\alpha(t)} - J_X = 0.$$

Their explicit solutions at an arbitrary time t are respectively given by

$$c_t^* = (J_X)^{1/(\Gamma-1)} = \frac{X_t + b(t)}{n(T) + \int_t^T H(v)n(v)dv}, \quad \theta_t^* = \frac{-J_X(\mu_S - r)}{J_{XX} \sigma_S^2} = -\frac{(X_t + b(t))(\mu_S - r)}{\sigma_S^2(\Gamma-1)},$$

$$\text{and } \xi_t^* = \alpha(t) \left(\frac{J_X \alpha(t)}{\lambda(t)} \right)^{1/(\Gamma-1)} - \alpha(t)X_t = \frac{(X_t + b(t))(H(t)-1)}{n(T) + \int_t^T H(v)n(v)dv} - \alpha(t)X_t$$

where

$$J(t, X_t) = U(X_t + b(t)) \left[n(T) + \int_t^T H(v)n(v)dv \right]^{1-\Gamma}, \quad b(t) \equiv \int_t^T I^* e^{g v - \int_t^v (r + \alpha(u))du} dv,$$

$$n(s) \equiv e^{-\int_t^s N(u)du}, \quad H(v) \equiv 1 + \frac{(\alpha(v))^{\Gamma/(\Gamma-1)}}{(\lambda(v))^{1/(\Gamma-1)}}, \quad \text{and} \quad N(u) \equiv \frac{\lambda(u) + \rho}{1-\Gamma} - \frac{(r + \alpha(u))\Gamma}{1-\Gamma} - \frac{(\mu_S - r)^2 \Gamma}{2\sigma_S^2(1-\Gamma)^2}.$$

The formulae for optimal wealth allocation strategies provided by Proposition 3 facilitate our numerical analysis. In subsequent sections, we term the function $b(\cdot)$ as human capital value by following Chen et al. (2006). The role of this function is to provide a wage-based channel through which human capital factors influence household life insurance demand and wealth allocation.

4. Model Implications in Numerical Analysis

We conduct numerical computation to examine the implications of human capital factors for household life insurance demand. These factors include human capital quality $\mathcal{A}$, substitutability β (pay-performance ratio), performance risk σ_A , and depreciation μ_A .

4.1. Parameter Calibration

We calibrate baseline parameters at the values that roughly reflect a typical U.S. corporation and household. Initial human capital quality $\mathcal{A}$ and the industrial benchmark of technology A are normalized to be 10 and 1, respectively. The input unit cost δ , initial wealth $\mathbf{x}$, retirement date T , the initial average product demand per customer Q , and the pay-performance ratio β are chosen at \$1, \$1000, 40 years, 1000, and 50%, respectively.^⑦ While these parameter choices can be motivated in more details, we omit these for brevity. Our results only vary quantitatively but not qualitatively with them.

Following Chu (2012), we choose the return-to-scale parameter γ at 0.6. We set the price elasticity of the customer base ε at 1.25. This is in the range of empirical estimates from Hoch et al. (1995) and Foster et al. (2008). Because human capital performance risk σ_A corresponds to technical shocks to productivity in Miao (2005), we set $\sigma_A = 15\%$ by using Miao's calibration results. We borrow the estimation result from Dinerstein et al. (2022) and choose the depreciation rate of human capital μ_A at -4.3%. We calibrate product demand volatility σ_q and the product demand growth rate μ_q to be 27.8% and 4.4%, respectively. Given this set of parameters related to operating earnings, our model at optimization generates the implied growth rate and volatility of operating earnings that match corresponding empirical estimates or calibration results in prior studies. The earnings growth rate is roughly 2.5%, consistent with the target value used by Miao (2005). Earnings volatility equals 25.5%, close to the estimated volatility of firm-level cash flows (25.5%) found in Strebulaeu (2007).

^⑦ Kline et al. (2019) argue that workers capture about 30% of every dollar of surplus/earnings. In general, rent-sharing parameter estimates for employees are in the range from 20% to 25% (Abowd and Lemieux, 1993). While our choice for the pay-to-performance ratio (50%) seems somewhat high, such a choice does not manipulate our main results. As shown in Panel D of Figures 2 and 3, varying the pay-to-performance ratio from 20% to 80% does not reverse the life-cycle pattern of life insurance demand as well as the sign of its influence on endogenous wages.

Table 1: Parameter Definition and Baseline Values.

Parameter Values	Parameter Definition
$\mathcal{A} = 10$	Human capital initial quality
$\mu_A = -4.3\%$	Human capital depreciation rate
$\sigma_A = 15\%$	Human capital performance risk
$\beta = 50\%$	Pay-performance ratio (human capital substitutability)
$A = 1$	Industrial productivity technology benchmark
$\delta = \$1$	Input unit price
$\gamma = 0.6$	Return-to-scale parameter
$\varepsilon = 1.25$	Price elasticity of the customer base
$Q = 1000$	Initial product demand
$\mu_q = 4.4\%$	Product demand growth rate
$\sigma_q = 27.8\%$	Product demand volatility
$r = 2.66\%$	Riskless interest rate
$\mu_S = 8.41\%$	Market portfolio return rate
$\sigma_S = 17.98\%$	Market portfolio risk
$\Gamma = -3$	Risk-aversion parameter
$\rho = 3\%$	Subjective discount rate
$g = 3\%$	Annualized wage growth rate
$T = 40 \text{ years}$	Retirement Date
$x = \$1000$	Initial wealth
$\tilde{\lambda} = 0.5\%$	Initial hazard rate
$\hat{\lambda} = 0.1125\%$	Time sensitivity of the hazard rate
$\tilde{\alpha} = 10^{-8}$	Initial premium-insurance ratio
$\hat{\alpha} = 0.1125\%$	Sensitivity of the premium-insurance ratio to age

We next consider the calibration for parameters involving household wealth allocation. The growth rate of wages is chosen at 3%, consistent with the annualized income growth rate in Chen et al. (2022). As in Pliska and Ye (2007), we calibrate the risk aversion parameter Γ to be -3, the subjective discount rate ρ to be 3%, the coefficients in the hazard-rate function $(\tilde{\lambda}, \hat{\lambda})$ to be (0.005, 0.001125), and the coefficients in the specification of the premium-insurance ratio $(\tilde{\alpha}, \hat{\alpha})$ to be $(10^{-8}, 0.001125)$. We consider a small value of $\tilde{\alpha}$, which ensures that the household wage earner has sufficient incentives to buy life insurance (life insurance purchase is positive). Finally,

we set the annualized rate of return on diversified portfolio μ_S at 8.41%, the volatility rate σ_S at 17.98%, and the riskless interest rate r at 2.66%. These numbers are taken from the result of structural estimation relying on a consumption-based asset pricing model in Ju and Miao (2012). A summary of baseline parameter values is presented in Table 1.

4.2. Influence of Human Capital Factors on Life Insurance Demand

We now examine the implication of human capital factors for optimal life insurance demand. For facilitating our discussions on related implications, we perform a life-cycle simulation of life insurance demand dynamics by using the formulae for optimal wealth allocation strategies given in Proposition 3. Our simulations consist of 100,000 paths with a length of 50 years that manifest a

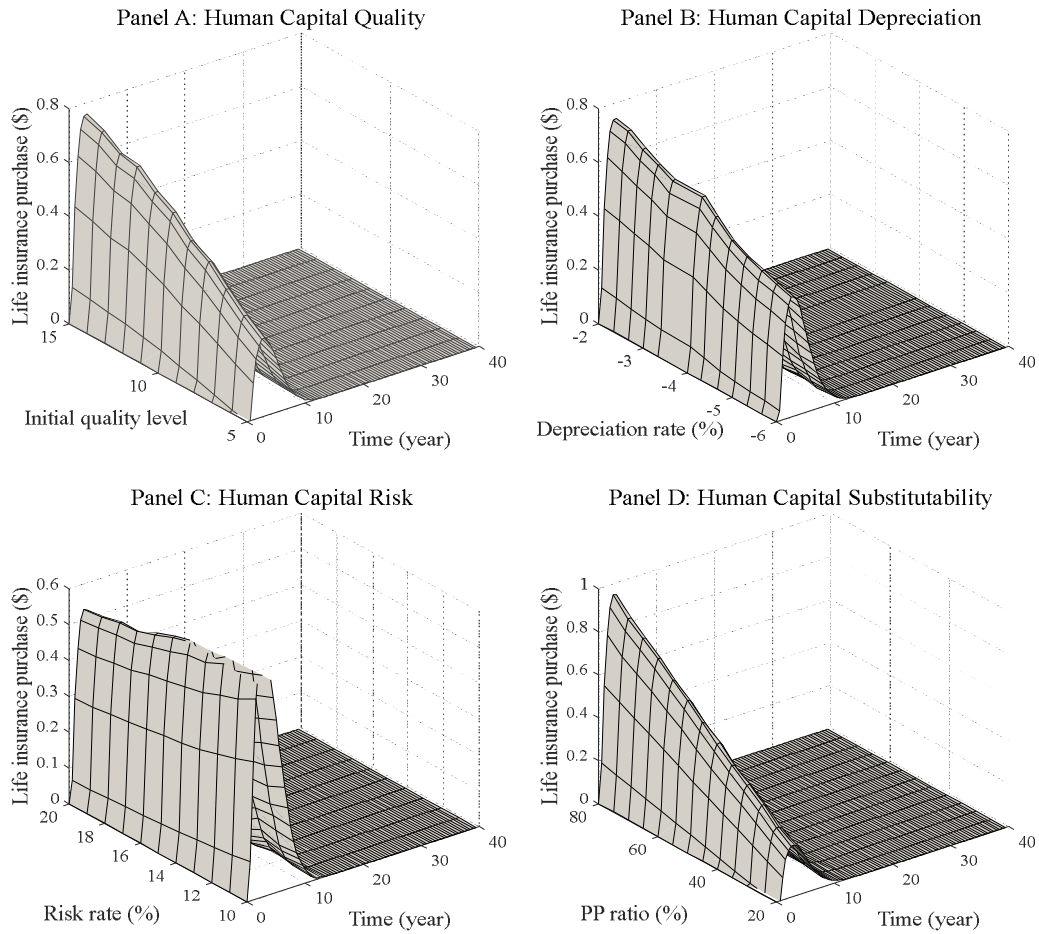


Figure 2: Dynamic Life Insurance Demand over Time and Human Capital Factors. In all Panels, the surfaces plot household life insurance purchase as a joint function of time and human capital factors. Except for the indicated parameters, model parameters are chosen at their baseline levels.

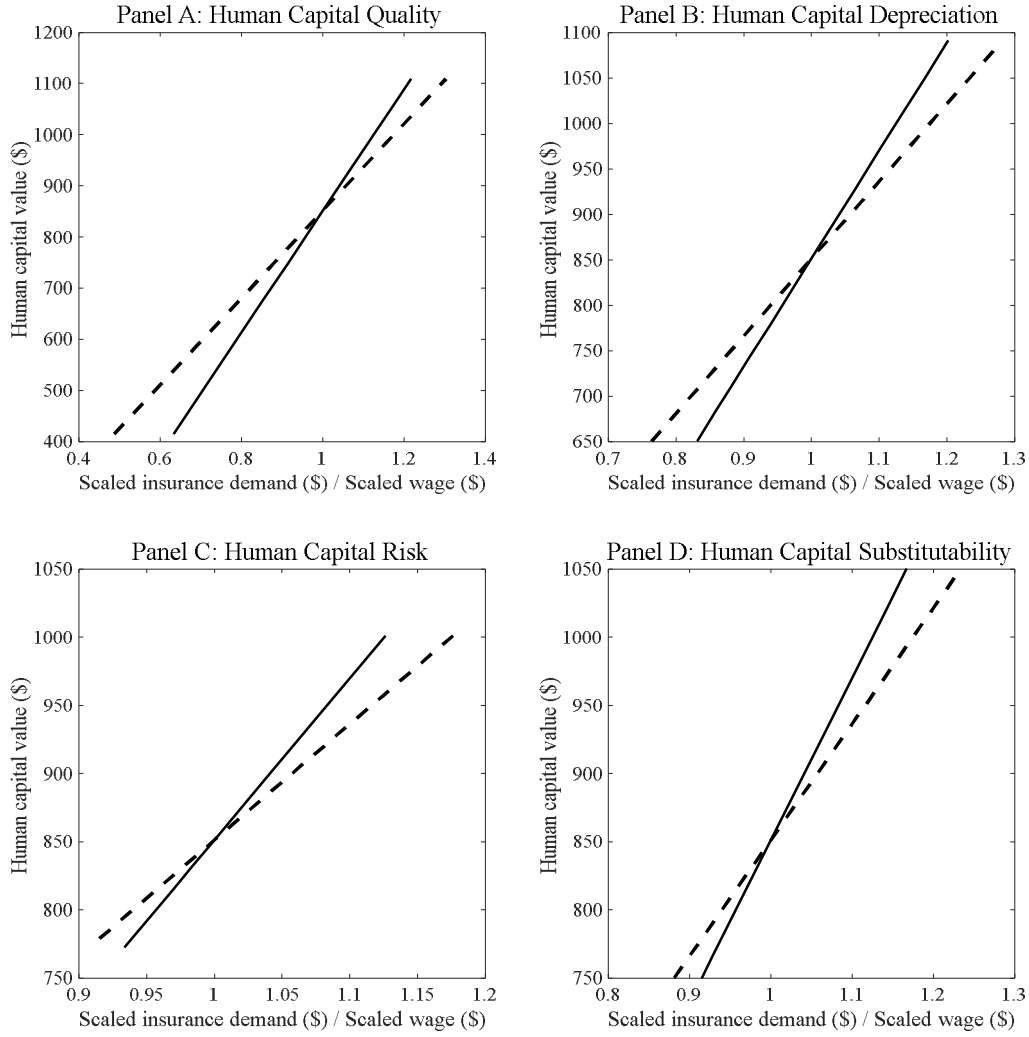


Figure 3: Interaction among Life Insurance Demand, Endogenous Wage, and Human Capital Value. In all Panels, the solid (dashed) lines depict the endogenous interactions between human capital value and optimal life insurance purchase at the initial time (initial wages) scaled by its baseline level. Human capital value is calculated using $b(0)$. The explicit form of this value function is given in Proposition 3. In Panels A, B, C, and D, we plot the lines by varying initial human capital quality A from 5 to 15, human capital depreciation rate μ_A from -6% to -2%, human capital risk σ_A from 10% to 20%, and the level of human capital substitutability β from 20% to 80%, respectively. Except for the indicated parameters, model parameters are set at their baseline levels.

reasonable retirement date. On the basis of the weekly simulation frequency, each of path contains 2600 simulated-observation points. Figure 2 plots the simulated life insurance purchase dynamics over time against different levels of human capital factors. Figure 3 depicts the endogenous interactions among human capital value, wages, and life insurance purchase by varying the levels of human capital factors. Figure 4 plots marginal firm value contributed by human capital $Perf(\cdot)$ against human capital factors. Several results from these figures immediately stand out.

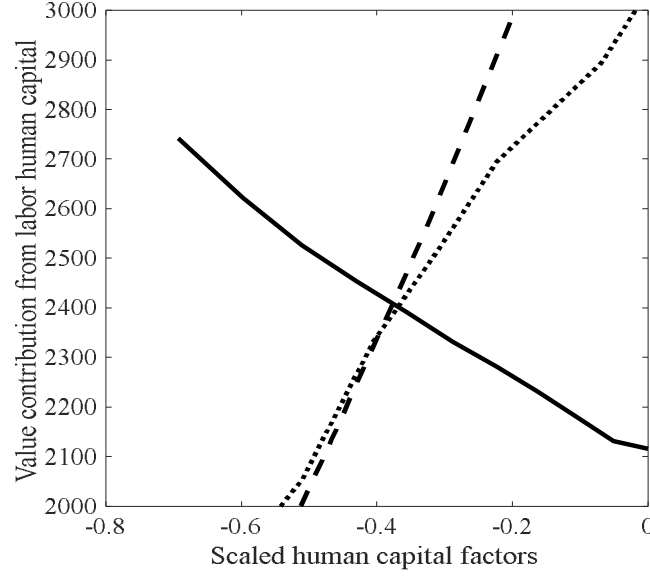


Figure 4: Human Capital Factors and Value Contribution from Human Capital. The dotted line depicts the value contribution from labor human capital $perf(\cdot)$ against the logarithm of scaled human capital quality (divided by 15) with varying initial human capital quality $\mathcal{A}$ from 5 to 15. The dashed line plots the value contribution from labor human capital against the human capital depreciation rate multiplied by 10 with varying the depreciation rate μ_A from -6% to -2%. The solid line plots the value contribution from human capital against the logarithm of scaled human capital risk (divided by 20%) with varying the risk rate σ_A from 10% to 20%. Except for the indicated parameters, model parameters are chosen at their baseline levels.

4.2.1. Human capital quality

We observe from Panel A in Figure 2 that households with a higher quality of human capital (or a higher productivity technology) will spend more in purchasing life insurance. Such a result is compatible with empirical evidence in Truett and Truett (1990) and Lee et al. (2010), which finds a robust positive association between household-level education and life insurance demand. The influence of human capital quality on life insurance demand can be decomposed into four parts:

$$\frac{\partial \xi^*}{\partial \mathcal{A}} = \frac{\partial \xi^*}{\partial b(\cdot)} \times \frac{\partial b(\cdot)}{\partial I^*} \times \frac{\partial I^*}{\partial Perf(\cdot)} \times \frac{\partial Perf(\cdot)}{\partial \mathcal{A}}.$$

The structure of this decomposition helps us clarify how changes in human capital quality affect life insurance demand through wages and human capital value. It is easily to find that the signs of all terms in this decomposition are strictly positive. This suggests that the wage earner possessing a higher productivity technology makes a greater contribution to firm value through production.

As a result, the firm pays more wages that implies a higher human capital value and motivates the household to spend more in purchasing life insurance. Such an implication can be corroborated by the patterns of Panel A in Figure 3 and of Figure 4, which reveal a positive interaction among life insurance demand, human capital value, wages, and human capital quality.

4.2.2. *Human capital depreciation*

Consider next the case of human capital depreciation. Contrary to the previous case in which human capital quality plays a positive factor for motivating life insurance purchase, human capital depreciation is a negative factor that discourages life insurance purchase. As Panel B in Figure 2 shows, household life insurance purchase is decreasing in the depreciation rate (a lower level of μ_A implies a larger depreciation rate). The mechanism through which changes in the depreciation rate affect life insurance demand is similar to the case of human capital quality. That is,

$$\frac{\partial \xi^*}{\partial \mu_A} = \frac{\partial \xi^*}{\partial b(\cdot)} \times \frac{\partial b(\cdot)}{\partial I^*} \times \frac{\partial I^*}{\partial Perf(\cdot)} \times \frac{\partial Perf(\cdot)}{\partial \mu_A}.$$

Although the signs of all terms in the decomposition above remain positive (for check these signs, please find the positive slope of all lines in Figure 3 and of the dashed line in Figure 4), the implications behind them are qualitatively different from the previous case. As mentioned earlier, a lower level of μ_A causes a larger depreciation rate. If the quality of human capital depreciates more quickly, the marginal increment in firm value generated by technological advantages over human capital becomes smaller. Given this fact, the firm sets a lower initial wage level according to the value-sharing rule, and thus, the household possessing less valuable human capital displays a weaker willingness to hedge mortality risk on human capital through life insurance. Our results provide empirical research a new testable implication on the negative association between human capital depreciation and life insurance demand. Several recent studies (e.g., Laureys, 2021; and Dinerstein et al., 2022) mainly concentrate on the externality from human capital depreciation in job creation and empirical estimation for the depreciation rate, but pay little attention to issues on household finance or wealth allocation.

4.2.3. Human capital risk

The existing literature on life insurance demand (e.g., Chen et al., 2022) considers the wage earner's mortality as the unique source of human capital risk, and reaches a consensus that hedging against wealth loss due to premature death may be a first-order motive for households to enter life insurance contracts. In our research, however, human capital risk does not necessarily become a motive for purchasing life insurance. It is observable that the household bearing a higher human capital risk spends less in purchasing life insurance (please find the surface of Panel C in Figure 2). Such a counterintuitive result is attributed to the fact that our wage model considers human capital performance as the source of risk, and can capture the role of human capital risk in determining equilibrium wages.[®] In contrast, prior studies assume that varying the mortality rate does not alter the level of household income.

Specifically, in our model, performance risk changes affect life insurance purchase through wages and human capital value. That means,

$$\frac{\partial \xi^*}{\partial \sigma_A} = \frac{\partial \xi^*}{\partial b(\cdot)} \times \frac{\partial b(\cdot)}{\partial I^*} \times \frac{\partial I^*}{\partial Perf(\cdot)} \times \frac{\partial Perf(\cdot)}{\partial \sigma_A},$$

where the first three terms on the right side of equation are consistently positive and the last term is negative (please find the patterns of Panel C in Figure 3 and the negative slope of the solid line in Figure 4). The key to generate the negative risk effect on insurance demand lies in the last term, which suggests that the contribution to firm value from labor human capital decreases with human capital performance risk. This is because operating earnings under optimal product pricing is an increasing concave function of productivity technology $\partial \hat{\pi}_t^* / \partial A_t = \frac{\varepsilon-1}{\eta} = \frac{1}{1-\gamma+(\varepsilon-1)^{-1}} < 1$.[®] Hence, the negative reactions of firm value to downward fluctuations in human capital performance are greater than the corresponding positive reactions to upward performance fluctuations. The impact

[®] Our model includes two human-capital-related risk sources. In addition to risk on human capital performance, we consider risk about premature death (mortality risk) by following prior studies. Our undisclosed numerical tests show a robust positive association between life insurance demand and premature death risk, consistent with prior findings.

[®] Using the condition of $\varepsilon \in (1, 1+\gamma^{-1})$, we can easily prove $(\varepsilon-1)^{-1} > \gamma$, which implies $1-\gamma+(\varepsilon-1)^{-1} > 1$.

of such asymmetries on firm value will be stronger if human capital performance is more volatile, and it further shapes a negative association between human capital risk and the value contribution from human capital. Given this fact, the firm tends to pay lower wages to a labor possessing more volatile human capital performance, thereby leading to a lower human capital value and a weaker demand for life insurance.

4.2.4. *Human capital substitutability*

We now move attention to the implication of human capital substitutability. In the model, we use the pay-performance ratio as an inverse measure of human capital substitutability, because the labor featuring scarce human capital typically has a strong power of bargaining for wages or holds many valuable outside job options. Panel D in Figure 2 indicates the positive influence of the pay-performance ratio on life insurance purchase, which means that households facing a higher degree of human capital substitutability are less willing to buy life insurance. It is worth noticing that the mechanism through which human capital substitutability affects life insurance demand lies in the value-sharing rule. In contrast, the mechanisms of the first three factors consistently rely on the marginal contribution to firm value from labor human capital.

The mechanism of human capital substitutability can be specified using

$$\frac{\partial \xi^*}{\partial \sigma_A} = \frac{\partial \xi^*}{\partial b(\cdot)} \times \frac{\partial b(\cdot)}{\partial I^*} \times \frac{\partial I^*}{\partial \beta},$$

where the terms on the right side of equation shows the positive interaction among human capital value, wages, and the pay-performance ratio (please find the positive slope of the lines in Panel D of Figure 3). In sum, a higher degree of human capital substitutability makes the household and firm agree on a lower pay-performance ratio that causes a lower wage level and a weaker demand for life insurance. Our results call for subsequent empirical tests, which provide a new direction to advance the literature on labor/worker substitutability and human capital scarcity (e.g., Horn and Wolinsky, 1988; Ciccone and Peri, 2005; Bovini and Paradisi, 2019; Huang and Tarkom, 2023; and Bils et al., 2024).

4.3. Additional Results

We additionally examine the implications of human capital factors for households' financial investment and consumption by plotting Figure 5. It is observable that the effect of human capital factors on investment/consumption is parallel to that on life insurance purchase. Improvements in human capital quality stimulate investment/consumption through human capital value and wages, while an increase in human capital substitutability, performance risk, or depreciation discourages investment/consumption through human capital value and wages (please find the patterns in Figure 5 and the positive slope of the lines in Figures 6-7).

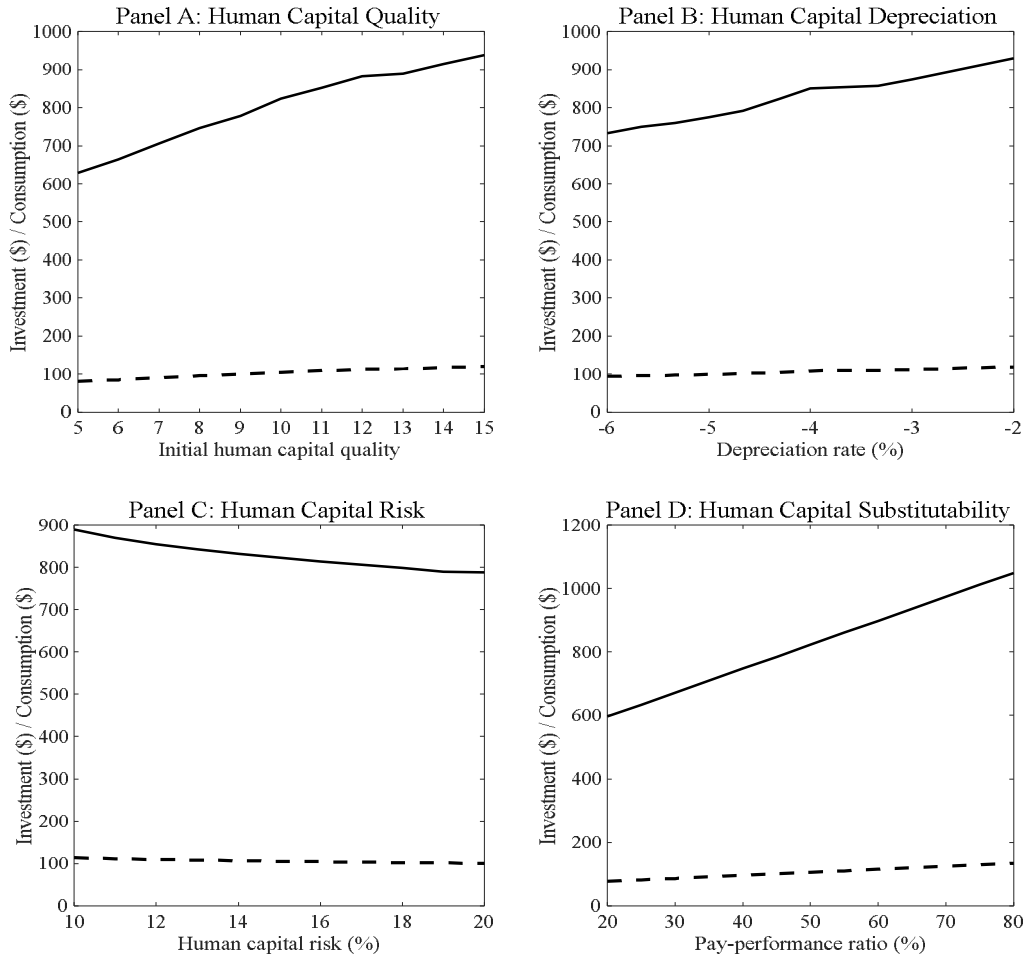


Figure 5: Household Investment, Consumption, and Human Capital Factors. The solid (dashed) lines plot household financial investment (consumption) against human capital factors. In Panels A, B, C, and D, we plot the lines by varying initial human capital quality A from 5 to 15, human capital depreciation rate μ_A from -6% to -2%, human capital risk σ_A from 10% to 20%, and the level of human capital substitutability β from 20% to 80%, respectively. Except for the indicated parameters, model parameters are set at their baseline levels.

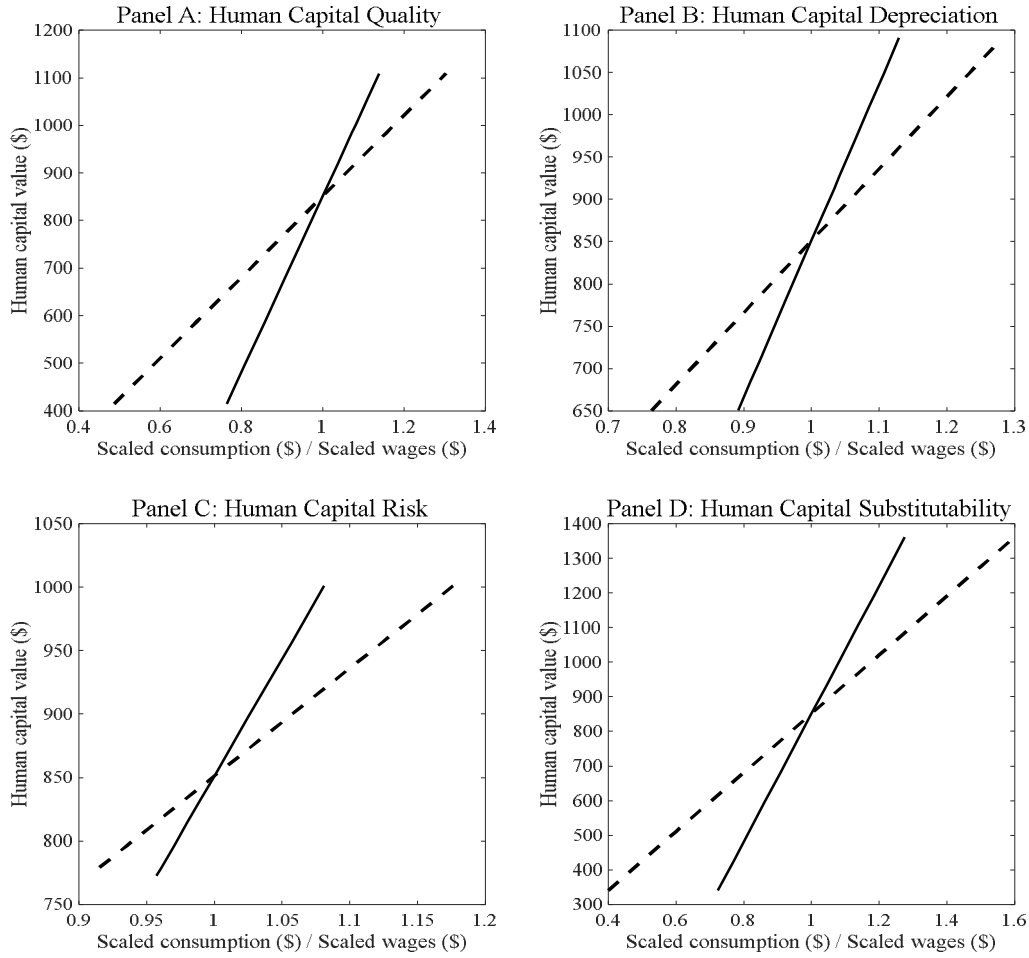


Figure 6: Interaction among Consumption, Endogenous Wages, and Human Capital Value. In all Panels, the solid (dashed) lines depict the endogenous interactions between human capital value and optimal consumption at the initial time (initial wages) scaled by its baseline level. Human capital value is calculated using $b(0)$. The explicit form of this value function is given in Proposition 3. In Panels A, B, C, and D, we plot the lines by varying initial human capital quality A from 5 to 15, human capital depreciation rate μ_A from -6% to -2%, human capital risk σ_A from 10% to 20%, and the level of human capital substitutability β from 20% to 80%, respectively. Except for the indicated parameters, model parameters are chosen at their baseline levels.

Our results offer a human-capital-based explanation for the stock market participation puzzle. This refers to a fact that despite the potential benefits of investing in stocks for long-term wealth growth, most of U.S. households are less willing to participate in the stock market or hold stocks (see Vissing-Jørgensen, 2002). We show that human capital performance risk, depreciation, and substitutability can be distinctive negative factors that discourage household financial investment by decreasing wages (or income). Moreover, our result of the positive influence of human capital quality on household financial investment is supported by empirical evidence in Bertaut (1998),

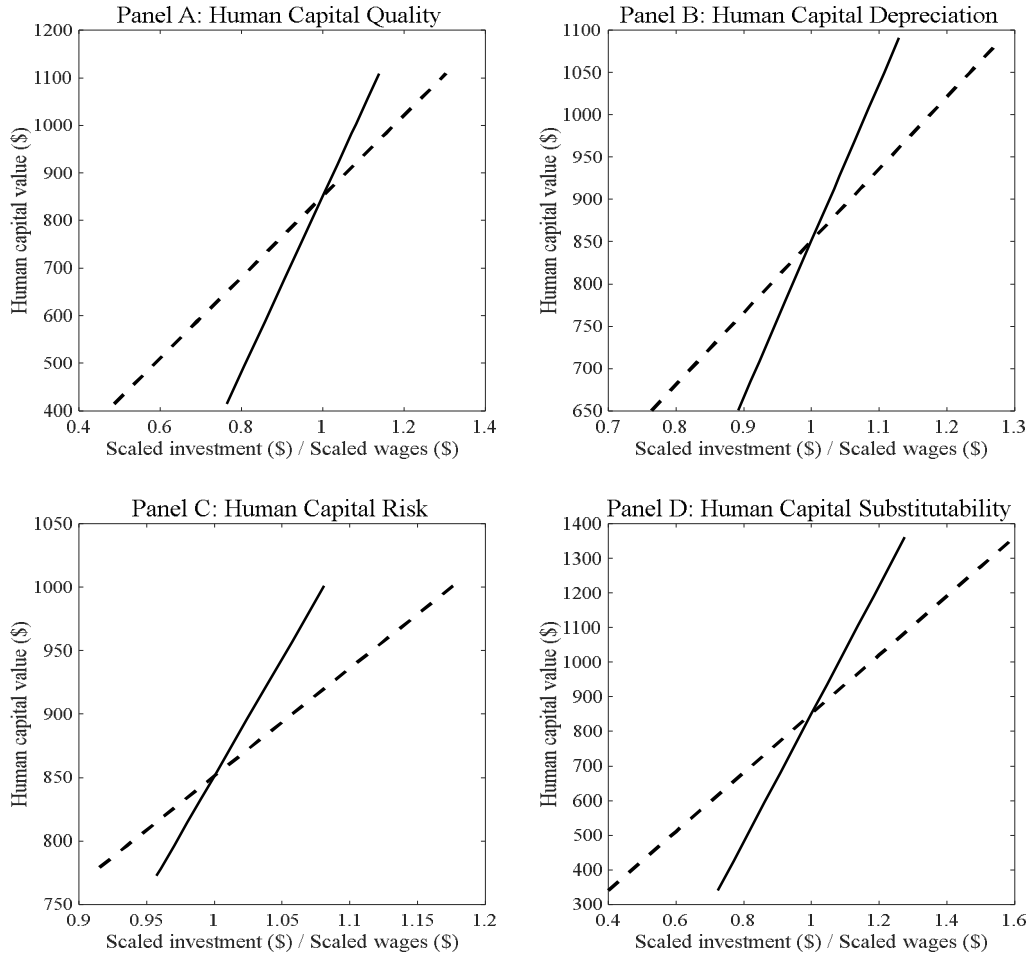


Figure 7: Interaction among Financial Investment, Endogenous Wages, and Human Capital Value. In all Panels, the solid (dashed) lines depict the endogenous interactions between human capital value and household investment at the initial time (initial wages) scaled by its baseline level. Human capital value is calculated using $b(0)$. The explicit form of this value function is given in Proposition 3. In Panels A, B, C, and D, we plot the lines by varying initial human capital quality A from 5 to 15, human capital depreciation rate μ_A from -6% to -2%, human capital risk σ_A from 10% to 20%, and the level of human capital substitutability β from 20% to 80%, respectively. Except for the indicated parameters, model parameters are chosen at their baseline levels.

which finds households consisting of better-educated members are more likely to hold stocks. Mu (2006) documents that income changes cause a weaker impact on consumption when households have members with higher education levels. This is compatible with the pattern of the dashed line in Panel A of Figure 5, which indicates that the positive reaction of consumption to improvements in human capital quality weakens as the level of quality rises (recall that changes in human capital quality affect consumption through endogenous wages as an increasing concave function of human capital quality).

4.4. A Summary of Theoretical Predictions for Subsequent Empirical Works

The proposed firm-household integrated model offers a rich set of new testable implications of human-capital-specific factors for households' optimal life insurance demand. For clarity, we give a summary of these implications in the following proposition:

Proposition 4. (Testable Implications) *Our model implications help us develop four hypotheses:*

- (i) heterogeneities in human-capital-specific factors are positively related to the across-household differences/variations in life insurance holding;*
- (ii) households hold more life insurance when their human capital has a higher quality;*
- (iii) households hold less life insurance when their human capital bears a higher performance risk, has a higher degree of substitutability, or depreciates more quickly;*
- (iv) human capital factors influence households' life insurance holding through their income.*

5. Conclusion

We construct a firm-based model of endogenous wage, and then integrate it into a standard framework of dynamic wealth allocation among consumption, financial investment, savings, and life insurance purchase. The structural form of endogenous wage, derived from the value-sharing rule between shareholders and the labor in place, consists of four human capital factors, including human capital quality, depreciation, performance risk, and substitutability. Endogenous wage can be a distinctive channel through which heterogeneities in human capital factors generate across-household differences in life insurance ownership. We find human capital quality to be a positive factor stimulating life insurance purchase, but human capital depreciation, risk, and substitutability are negative factors discouraging insurance purchase. When human capital possessed by the labor in place displays a high quality (provides unstable performance, depreciates quickly, and is highly substitutable), the expected value contribution generated by human capital is large (the expected value contribution is small, the pay-performance ratio is low) and firms set a high (low) level of wage according to the value-sharing rule. Given this fact, households correspondingly spend more (less) in purchasing life insurance.

Our model can be viewed as the first step toward bridging the gap between corporate finance and household life insurance demand theories (through endogenous wages). Using our modelling framework, subsequent works can advance the life-cycle theory of household insurance demand in many corporate-finance-related dimensions, such as debt financing, innovation, cash holding, etc. Also, our theoretical results offer empirical studies a useful guide to finding new human-capital-related determinants of household life insurance ownership.

Appendix A: Proof for Proposition 1

From expression (4) in Section 2.1.5, it is known that earnings take the form:

$$\hat{\pi}(p_t, Q_t, A_t) = p_t^{1-\varepsilon} Q_t - \delta p_t^{-\varepsilon/\gamma} Q_t^{1/\gamma} A_t^{-1/\gamma}.$$

Differentiating the earnings function with respect to product price yields the FOC:

$$\left. \frac{\partial \hat{\pi}}{\partial p_t} \right|_{p_t = p_t^*} = (1-\varepsilon)(p_t^*)^{-\varepsilon} Q_t + \frac{\delta \varepsilon}{\gamma} (p_t^*)^{-1-\varepsilon/\gamma} Q_t^{1/\gamma} A_t^{-1/\gamma} = 0,$$

which implies $p_t^* = Q_t^{(1-\gamma)/\eta} A_t^{-1/\eta} (\frac{\delta \varepsilon}{\varepsilon - \eta})^{\gamma/\eta}$. The optimal product price is strictly nonnegative, since $\eta = \varepsilon + \gamma - \varepsilon \gamma = \gamma + \varepsilon(1-\gamma) > 0$ and $\varepsilon - \eta = \gamma(\varepsilon - 1) > 0$. Differentiating the FOC with respect to product price yields the following derivation:

$$\begin{aligned} \left. \frac{\partial^2 \hat{\pi}}{\partial p_t^2} \right|_{p_t = p_t^*} &= -\varepsilon(1-\varepsilon)(p_t^*)^{-\varepsilon-1} Q_t - \frac{\delta \varepsilon(\varepsilon+\gamma)}{\gamma^2} (p_t^*)^{-2-\varepsilon/\gamma} Q_t^{1/\gamma} A_t^{-1/\gamma} \\ &= (\varepsilon-1)(p_t^*)^{-\varepsilon-1} Q_t \left[\varepsilon - (p_t^*)^{-\frac{\eta}{\gamma}} Q_t^{1/\gamma-1} A_t^{-1/\gamma} \frac{\delta \varepsilon}{\gamma} \frac{1+\varepsilon/\gamma}{\varepsilon-1} \right] \\ &= (\varepsilon-1)(p_t^*)^{-\varepsilon-1} Q_t \left[\varepsilon - (p_t^*)^{-\frac{\eta}{\gamma}} (p_t^*)^{\frac{\eta}{\gamma}} \frac{\gamma+\varepsilon}{\gamma} \right] \\ &= (\varepsilon-1)(p_t^*)^{-\varepsilon-1} Q_t (\varepsilon-1-\varepsilon \gamma^{-1}). \end{aligned}$$

Because $\varepsilon \in (1, 1+\gamma^{-1})$, we have $\varepsilon-1 < \gamma^{-1} < \varepsilon \gamma^{-1}$ and the SOC

$$\left. \frac{\partial^2 \hat{\pi}}{\partial p_t^2} \right|_{p_t = p_t^*} = \underbrace{(\varepsilon - 1)(p_t^*)^{-\varepsilon - 1} Q_t}_{\text{sign}(+)} \underbrace{(\varepsilon - 1 - \varepsilon \gamma^{-1})}_{\text{sign}(-)} < 0.$$

To further verify the optimality of product pricing strategies, we provide a simple numerical example of testing the formula for optimal price, SOC, FOC, and operating earnings under product pricing optimization. The related results are presented in Table A. Observe that the optimal prices obtained from numerical search are close to those calculated from our formula. All the values of FOC at the optimal price approach zero. The signs of SOC at the optimal price remain negative across various parameter combinations. These results corroborate optimality implied by product pricing, and help complete the proof for Proposition 1.

Table A: Numerical Verification for Product Pricing Optimization

This table presents the results of numerical verification for product pricing optimization. The columns (from left to right) report the optimal product price obtained from numerical search, the optimal product price calculated from the formula in Proposition 1, the first-order-condition (from Proposition 1) at optimization, the second-order condition (from Proposition 1) at optimization, operating earnings under optimal product pricing solved from numerical search, and operating earnings under optimal product pricing obtained from Proposition 1, respectively.

Parameter Choices	Optimal Price (1)	Optimal Price (2)	F.O.C.	S.O.C.	Operating Earnings (1)	Operating Earnings (2)
Baseline	\$39.1899	\$39.1898	-2*10 ⁻¹⁵	-0.1193	\$351.7136	\$351.7136
$Q = 500$	\$30.4585	\$30.4584	-1*10 ⁻¹⁵	-0.1051	\$187.2947	\$187.2947
$Q = 1500$	\$45.4157	\$45.4158	-2*10 ⁻¹⁵	-0.1284	\$508.4780	\$508.4780
$\varepsilon = 1.1$	\$76.3202	\$76.3201	-2*10 ⁻¹⁶	-0.0193	\$612.8818	\$612.8818
$\varepsilon = 1.5$	\$22.3607	\$22.3606	3*10 ⁻¹⁵	-0.4229	\$169.1794	\$169.1794
$\gamma = 0.5$	\$59.9483	\$59.9484	2*10 ⁻¹⁶	-0.0563	\$323.4432	\$323.4432
$\gamma = 0.7$	\$24.7297	\$24.7297	2*10 ⁻¹⁵	-0.2815	\$385.6503	\$385.6503
$\delta = \$0.5$	\$26.8519	\$26.8519	-1*10 ⁻¹⁵	-0.2792	\$386.5794	\$386.5794
$\delta = \$2$	\$57.1967	\$59.1968	-8*10 ⁻¹⁶	-0.0509	\$319.9924	\$319.9924

Appendix B: Proof for Proposition 2

Consider first the derivation of $Perf(\cdot)$. We have known from Proposition 1 and (5) that

$$Perf(0; Q, \mathcal{A}) = \mathbb{E}_0 \int_0^{T \wedge \tau} (\hat{\pi}^*(Q_s, A_s) - \hat{\pi}^*(Q_s, A)) e^{-rs} ds = A - B$$

$$A \equiv \mathbb{E}_0 \int_0^\infty (\hat{\pi}^*(Q_s, A_s) - \hat{\pi}^*(Q_s, A)) e^{-rs} ds, \quad B \equiv \mathbb{E}_0 \int_{T \wedge \tau}^\infty (\hat{\pi}^*(Q_s, A_s) - \hat{\pi}^*(Q_s, A)) e^{-rs} ds.$$

Using Ito's lemma and Proposition 1 yields

$$d\hat{\pi}^*(Q_t, A_t) = \mu_\pi \hat{\pi}^*(Q_t, A_t) dt + \sigma_{\pi q} \hat{\pi}^*(Q_t, A_t) db_t + \sigma_{\pi A} \hat{\pi}^*(Q_t, A_t) dw_t,$$

$$\text{and } d\hat{\pi}^*(Q_t, A) = \mu_{\pi A} \hat{\pi}^*(Q_t, A) dt + \sigma_{\pi q} \hat{\pi}^*(Q_t, A) db_t$$

where $\mu_{\pi A} \equiv \mathbb{E}(\eta, \mu_q, \sigma_q)$, $\mu_\pi \equiv \mathbb{E}(\eta(\varepsilon - 1)^{-1}, \mu_A, \sigma_A) + \mu_{\pi A}$, $\sigma_{\pi A} \equiv \mathbb{E}(\eta(\varepsilon - 1)^{-1}, \sigma_A, 0)$, and $\sigma_{\pi q} \equiv \mathbb{E}(\eta, \sigma_q, 0)$. Hence, according to equations (1)-(2) in Goldstein et al. (2001), we can derive the term A as the following:

$$\begin{aligned} A &= \mathbb{E}_0 \int_0^\infty \hat{\pi}^*(Q_s, A_s) e^{-rs} ds - \mathbb{E}_0 \int_0^\infty \hat{\pi}^*(Q_s, A) e^{-rs} ds \\ &= \frac{\hat{\pi}^*(Q, \mathcal{A})}{r - \mu_\pi} - \frac{\hat{\pi}^*(Q, A)}{r - \mu_{\pi A}} \\ &= \frac{A(Q, \mathcal{A}) \Phi}{r - \mathbb{E}(\eta(\varepsilon - 1)^{-1}, \mu_A, \sigma_A) - \mathbb{E}(\eta, \mu_q, \sigma_q)} - \frac{A(Q, A) \Phi}{r - \mathbb{E}(\eta, \mu_q, \sigma_q)} \\ &= A(Q, \mathcal{A}) K - A(Q, A) G = B(Q, \mathcal{A}, Q). \end{aligned} \tag{A1}$$

We then rearrange the term B as the following:

$$\begin{aligned} B &= \mathbb{E}_0 \left[(1_{(\tau > T)} + 1_{(\tau \leq T)}) \int_{T \wedge \tau}^\infty (\hat{\pi}^*(Q_s, A_s) - \hat{\pi}^*(Q_s, A)) e^{-rs} ds \right] = B1 + B2 \\ B1 &\equiv \mathbb{E}_0 1_{(\tau > T)} \int_T^\infty (\hat{\pi}^*(Q_s, A_s) - \hat{\pi}^*(Q_s, A)) e^{-rs} ds \\ \text{and } B2 &\equiv \mathbb{E}_0 1_{(\tau \leq T)} \int_\tau^\infty (\hat{\pi}^*(Q_s, A_s) - \hat{\pi}^*(Q_s, A)) e^{-rs} ds. \end{aligned}$$

Similar to the derivation for (A1), we use Tower law and further derive the above two terms as:

$$\begin{aligned}
B1 &= \mathbb{E}_0 \left[1_{(\tau > T)} e^{-rT} \mathbb{E}_T \int_T^\infty (\hat{\pi}^*(Q_s, A_s) - \hat{\pi}^*(Q_s, \mathcal{A})) e^{-r(s-T)} ds \right] \\
&= \mathbb{E}_0 \left[1_{(\tau > T)} e^{-rT} (\mathbb{A}(Q_T, A_T) K - \mathbb{A}(Q_T, \mathcal{A}) G) \right] \\
&= F(T, 0) e^{-rT} \mathbb{E}_0 \left[Q_T^{1/\eta} \mathbb{B}(1, A_T, 1) \right] \\
&= F(T, 0) e^{-rT} \int_{-\infty}^\infty \int_{-\infty}^\infty f_0^{\ln Q}(s, T) f_0^{\ln A}(u, T) \mathbb{B}(1, e^u, 1) e^{s/\eta} ds du \\
&= F(T, 0) e^{-rT} \int_{-\infty}^\infty \int_{-\infty}^\infty \mathbb{C}(s, u, T) e^{s/\eta} ds du \tag{A2}
\end{aligned}$$

and

$$\begin{aligned}
B2 &= \mathbb{E}_0 \left[1_{(\tau \leq T)} e^{-r\tau} \mathbb{E}_\tau \int_\tau^\infty (\hat{\pi}^*(Q_s, A_s) - \hat{\pi}^*(Q_s, \mathcal{A})) e^{-r(s-\tau)} ds \right] \\
&= \mathbb{E}_0 \left[1_{(\tau \leq T)} e^{-r\tau} (\mathbb{A}(Q_\tau, A_\tau) K - \mathbb{A}(Q_\tau, \mathcal{A}) G) \right] \\
&= \mathbb{E}_0 \left[1_{(\tau \leq T)} e^{-r\tau} Q_\tau^{1/\eta} \mathbb{B}(1, A_\tau, 1) \right] \\
&= \int_{0_+}^T \int_{-\infty}^\infty \int_{-\infty}^\infty f(v, 0) f_0^{\ln Q}(s, v) f_0^{\ln A}(u, v) \mathbb{B}(1, e^u, 1) e^{s/\eta - rv} ds du dv \\
&= \int_{0_+}^T \int_{-\infty}^\infty \int_{-\infty}^\infty f(v, 0) \mathbb{C}(s, u, v) e^{s/\eta - rv} ds du dv \tag{A3}
\end{aligned}$$

where $f_0^{\ln Q}(s, v)$ is the density function of $\ln Q_v = s$ conditional on $\mathcal{F}_0$, $f_0^{\ln A}(u, v)$ is the density function of $\ln A_v = u$ conditional on $\mathcal{F}_0$, and $f(v, 0)$ is the density function of death $\tau = v$ conditional on $\mathcal{F}_0$. These three random variables are mutually independent. The forms of these density functions have been given in Section 2.2.1 and Proposition 2. Using (A1)-(A3) has

$$\begin{aligned}
Perf(0; Q, \mathcal{A}) &= \mathbb{B}(Q, \mathcal{A}, Q) - F(T, 0) e^{-rT} \int_{-\infty}^\infty \int_{-\infty}^\infty \mathbb{C}(s, u, T) e^{s/\eta} ds du \\
&\quad - \int_{0_+}^T \int_{-\infty}^\infty \int_{-\infty}^\infty f(v, 0) \mathbb{C}(s, u, v) e^{s/\eta - rv} ds du dv.
\end{aligned}$$

Consider next the derivation for VF . From Section 3.1, we have known that its expression:

$$VF \equiv \mathbb{E}_0 \int_0^{T \wedge \tau} e^{(g-r)s} ds. \text{ It can be further derived as the following:}$$

$$\begin{aligned}
VF &= \mathbb{E}_0 \left[(1_{(\tau > T)} + 1_{(\tau \leq T)}) \int_0^{T \wedge \tau} e^{(g-r)s} ds \right] \\
&= \mathbb{E}_0 \left[1_{(\tau > T)} \int_0^T e^{(g-r)s} ds \right] + \mathbb{E}_0 \left[1_{(\tau \leq T)} \int_0^\tau e^{(g-r)s} ds \right] \\
&= F(T, 0) \int_0^T e^{(g-r)s} ds + \int_{0+}^T f(s, 0) \int_0^s e^{(g-r)u} du ds \\
&= F(T, 0) (e^{(g-r)T} - 1) (g-r)^{-1} + \int_{0+}^T f(s, 0) (e^{(g-r)s} - 1) (g-r)^{-1} ds \\
&= \left[\mathbb{D}(T) F(T, 0) + \int_{0+}^T \mathbb{D}(s) f(s, 0) ds \right] (g-r)^{-1}.
\end{aligned}$$

This completes the proof.

Appendix C: Proof for Proposition 3

We make the proof for Proposition 3 in two steps: Step 1 derives the HJB equation. Step 2 solves optimal wealth allocation strategies from HJB equation. We first define $Y_t^* \equiv Y(\xi_t^*)$ and

$$\Psi_t \equiv \int_0^t (F(s, 0)U(c_s^*) + f(s, 0)U(Y_s^*)) e^{-\rho s} ds + J(t, X_t) F(t, 0) e^{-\rho t}.$$

It is easily to show that this process is a martingale, because

$$\begin{aligned}
\Psi_0 &= J(0, \mathbf{x}) \\
&= \max_{(c, \xi, \theta)} \mathbb{E}_0 \left[\int_0^T (F(s, 0)U(c_s) + f(s, 0)U(Y_s)) e^{-\rho s} ds + F(T, 0)U(X_T) e^{-\rho T} \right] \\
&= \max_{(c, \xi, \theta)} \mathbb{E}_0 \left[\mathbb{E}_t \left[\int_t^T (F(s, 0)U(c_s) + f(s, 0)U(Y_s)) e^{-\rho s} ds + F(T, 0)U(X_T) e^{-\rho T} \right] \right. \\
&\quad \left. + \int_0^t (F(s, 0)U(c_s) + f(s, 0)U(Y_s)) e^{-\rho s} ds \right] \\
&= \max_{(c, \xi, \theta)} \mathbb{E}_0 \left[F(t, 0) e^{-\rho t} \mathbb{E}_t \left[\int_t^T (F(s, t)U(c_s) + f(s, t)U(Y_s)) e^{-\rho(s-t)} ds + F(T, t)U(X_T) e^{-\rho(T-t)} \right] \right. \\
&\quad \left. + \int_0^t (F(s, 0)U(c_s) + f(s, 0)U(Y_s)) e^{-\rho s} ds \right] \\
&= \mathbb{E}_0 \left[\int_0^t (F(s, 0)U(c_s^*) + f(s, 0)U(Y_s^*)) e^{-\rho s} ds + J(t, X_t) F(t, 0) e^{-\rho t} \right] \\
&= \mathbb{E}_0 [\Psi_t].
\end{aligned}$$

Using the martingale property $\mathbb{E}_0 [d\Psi_t] = 0$ yields the so-called HJB equation:

$$J_t + J_X [rX - c^* - \xi^* + I^* + \theta^*(\mu_S - r)] + 0.5 J_{XX} (\theta^*)^2 \sigma_S^2 + \lambda(t)U(Y^*) + U(c^*) - [\lambda(t) + \rho]J = 0,$$

which is equivalent to

$$\max_{c, \xi, \theta} \left\{ J_t + J_X [rX - c - \xi + I + \theta(\mu_S - r)] + 0.5 J_{XX} \theta^2 \sigma_S^2 + \lambda(t)U(Y) + U(c) - [\lambda(t) + \rho]J \right\} = 0.$$

According to the HJB equation above, we show the first-order-conditions as the following:

$$U_c(c^*) - J_X = 0, \quad \theta^* J_{XX} \sigma_S^2 + J_X (\mu_S - r) = 0, \quad \text{and} \quad \frac{\lambda(t)U_Y(Y(\xi^*))}{\alpha(t)} - J_X = 0.$$

From these conditions, we derive optimal allocation strategies as

$$c_t^* = (J_X)^{1/(\Gamma-1)}, \quad \theta_t^* = \frac{-J_X (\mu_S - r)}{J_{XX} \sigma_S^2}, \quad \text{and} \quad \xi_t^* = \alpha(t) \left(\frac{J_X \alpha(t)}{\lambda(t)} \right)^{1/(\Gamma-1)} - \alpha(t) X_t. \quad (\text{A4})$$

Plugging (c^*, θ^*, ξ^*) into the HJB equation yields

$$J_X [(r + \alpha(t))X + I^*] + J_X^{\Gamma/(\Gamma-1)} (\Gamma^{-1} - 1)H(t) - 0.5 J_X^2 (\mu_S - r)^2 J_{XX}^{-1} \sigma_S^{-2} + J_t - [\lambda(t) + \rho]J = 0.$$

To solve the HJB equation, we use a trial solution: $J(t, X_t) = a(t)U(X_t + b(t))$ subject to the boundary conditions of $a(T) = 1$ and $b(T) = 0$. Then the HJB equation can be rewritten as:

$$\begin{aligned} & a(t)(X_t + b(t))^{\Gamma-1} [(r + \alpha(t))X + I^*] - 0.5 a(t)(X_t + b(t))^\Gamma (\mu_S - r)^2 \sigma_S^{-2} (\Gamma - 1)^{-1} \\ & + a(t)^{\Gamma/(\Gamma-1)} (X_t + b(t))^\Gamma (\Gamma^{-1} - 1)H(t) + a'(t)U(X_t + b(t)) + a(t)(X_t + b(t))^{\Gamma-1} b'(t) \\ & - [\lambda(t) + \rho]a(t)U(X_t + b(t)) = 0 \end{aligned}$$

or

$$\begin{aligned} & a(t)(X_t + b(t))^{-1} [(r + \alpha(t))X + I^*] - 0.5 a(t)(\mu_S - r)^2 \sigma_S^{-2} (\Gamma - 1)^{-1} \\ & + a(t)^{\Gamma/(\Gamma-1)} (\Gamma^{-1} - 1)H(t) + a'(t)\Gamma^{-1} + a(t)(X_t + b(t))^{-1} b'(t) - [\lambda(t) + \rho]a(t)\Gamma^{-1} = 0. \end{aligned}$$

We also consider a trial solution to $b(t)$, That is,

$$b(t) = \int_t^T I_v^* e^{-\int_t^v (r+\alpha(u)) du} dv = \int_t^T I^* e^{g v - \int_t^v (r+\alpha(u)) du} dv, \quad (A5)$$

which satisfies $b'(t) = (r + \alpha(t))b(t) - I_t^*$ and $b(T) = 0$ simultaneously. This trial solution can help us reduce the HJB equation to an ordinary differential equation (ODE):

$$\begin{aligned} & a(t)(r + \alpha(t)) - 0.5 a(t)(\mu_s - r)^2 \sigma_s^{-2} (\Gamma - 1)^{-1} + a(t)^{\Gamma/(\Gamma-1)} (\Gamma^{-1} - 1) H(t) + a'(t) \Gamma^{-1} \\ & - [\lambda(t) + \rho] a(t) \Gamma^{-1} = 0, \end{aligned}$$

which can be further reduced to

$$\begin{aligned} & (r + \alpha(t)) - 0.5 (\mu_s - r)^2 \sigma_s^{-2} (\Gamma - 1)^{-1} + a(t)^{1/(\Gamma-1)} (\Gamma^{-1} - 1) H(t) + a'(t) \Gamma^{-1} (a(t))^{-1} \\ & - [\lambda(t) + \rho] \Gamma^{-1} = 0. \end{aligned}$$

The ODE above admits a close form solution. To find it, we use a trial solution:

$$a(t) = (e(t))^{1-\Gamma} \quad \text{s.t.} \quad e(T) = 1.$$

Because $a'(t) = a(t)(1-\Gamma)(e(t))^{-1} e'(t)$ and $a(t)^{1/(\Gamma-1)} = (e(t))^{-1}$, we can rewrite the ODE

$$\begin{aligned} & (r + \alpha(t)) - 0.5 (\mu_s - r)^2 \sigma_s^{-2} (\Gamma - 1)^{-1} + (e(t))^{-1} (\Gamma^{-1} - 1) H(t) + \Gamma^{-1} (1-\Gamma) (e(t))^{-1} e'(t) \\ & - [\lambda(t) + \rho] \Gamma^{-1} = 0 \end{aligned}$$

Given $N(t) \equiv \frac{\lambda(t) + \rho}{1-\Gamma} - \frac{(r + \alpha(t))\Gamma}{1-\Gamma} - \frac{(\mu_s - r)^2 \Gamma}{2\sigma_s^2 (1-\Gamma)^2}$, the ODE thus can be rearranged as

$$H(t) + e'(t) - e(t)N(t) = 0,$$

which has its close form solution

$$\begin{aligned} e(t) &= e^{-\int_t^T N(u) du} + \int_t^T H(v) e^{-\int_t^v N(u) du} dv = n(T) + \int_t^T H(v) n(v) dv \\ \text{and } a(t) &= \left[n(T) + \int_t^T H(v) n(v) dv \right]^{1-\Gamma}. \end{aligned} \quad (A6)$$

Hence, using (A4), (A5), (A6), and the trial solution to the indirect utility function, we have

$$J(t, X_t) = U(X_t + b(t)) \left[n(T) + \int_t^T H(v) n(v) dv \right]^{1-\Gamma}, \quad b(t) \equiv \int_t^T I^* e^{g v - \int_t^v (r + \alpha(u)) du} dv,$$

$$c_t^* = (J_X)^{1/(\Gamma-1)} = \frac{X_t + b(t)}{n(T) + \int_t^T H(v) n(v) dv}, \quad \theta_t^* = \frac{-J_X(\mu_S - r)}{J_{XX} \sigma_S^2} = -\frac{(X_t + b(t))(\mu_S - r)}{\sigma_S^2(\Gamma - 1)},$$

$$\text{and } \xi_t^* = \alpha(t) \left(\frac{J_X \alpha(t)}{\lambda(t)} \right)^{1/(\Gamma-1)} - \alpha(t) X_t = \frac{(X_t + b(t))(H(t) - 1)}{n(T) + \int_t^T H(v) n(v) dv} - \alpha(t) X_t.$$

This completes the proof.

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